In general

$$\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + cr^3 + \cdots = \frac{c}{1-r}$$

c: any constant

$|r| < 1$

$$S_N = \sum_{n=0}^{N} cr^n = \frac{c(1-r^{N+1})}{1-r}$$

Example: Evaluate $\sum_{n=0}^{\infty} 3^{-n}$

Soln: $S = \sum_{n=0}^{\infty} \underbrace{\left(\frac{1}{3}\right)^n}_{r^n} = \frac{1}{1-r} = \frac{1}{1-1/3} = 3/2$

Example: Evaluate $\sum_{n=3}^{\infty} 7(-3/4)^n$

Solution: $S = \sum_{n=3}^{\infty} \underbrace{7(-3/4)^n}_{ar^n}$

careful! The sum is from $n=3$

So $S=\sum_{n=3}^{\infty} 7\left(-\frac{3}{4}\right)^n$

How do we fix this?

Let $m=n-3 \Rightarrow n=m+3$

$$S=\sum_{m+3=3}^{\infty} 7\left(-\frac{3}{4}\right)^{m+3} = \sum_{m=0}^{\infty} \underbrace{7\left(-\frac{3}{4}\right)^3}_{c}\underbrace{\left(-\frac{3}{4}\right)^m}_{r^m}$$

$$= 7\left(-\frac{3}{4}\right)^3 \cdot \frac{1}{1-\left(-\frac{3}{4}\right)} = 7\left(-\frac{3}{4}\right)^3 \frac{4}{7} = -\frac{27}{16}$$

In general:

$$\sum_{n=M}^{\infty} cr^n = \frac{cr^M}{1-r}$$

Divergence Test

Recall that $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \underbrace{\sum_{n=1}^{N} a_n}_{S_N}$

Now, suppose that $S=\lim_{n\to\infty} S_N$ is finite (i.e. $\sum_{n=1}^{\infty} a_n$ converges)

We know: $S_N = \sum_{n=1}^{N} a_n$ and $S_{N-1} = \sum_{n=1}^{N-1} a_n$

$$\Rightarrow S_N - S_{N-1} = \sum_{n=1}^{N} a_n - \sum_{n=1}^{N-1} a_n = a_N$$

So $\lim_{N\to\infty} a_N = \underbrace{\lim_{N\to\infty} S_N}_{S} - \underbrace{\lim_{N\to\infty} S_{N-1}}_{S} = 0$

We just showed that if $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$

So:

If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges

Divergence test

Example: Show that $\sum_{n=1}^{\infty} \frac{n^2+1}{3n^2}$ diverges

Soln: Since $\lim_{n\to\infty} \frac{n^2+1}{3n^2} = \frac{1}{3} \neq 0$

$\sum_{n=1}^{\infty} \frac{n^2+1}{3n^2}$ diverges by the divergence test.

Warning: If $\lim_{n\to\infty} a_n = 0$ this does not mean that $\sum_{n=1}^{\infty} a_n$ necessarily converges

(It could either converge or diverge)

Example $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (even though $\frac{1}{\sqrt{n}} \to 0$)

because $S_N = \sum_{n=1}^{N} \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{N}}$

$$\geq \frac{1}{\sqrt{N}} + \frac{1}{\sqrt{N}} + \cdots + \frac{1}{\sqrt{N}}$$

$$= \sqrt{N}$$

So $S_N \geq \sqrt{N} \Rightarrow \lim_{N\to\infty} S_N = \infty$

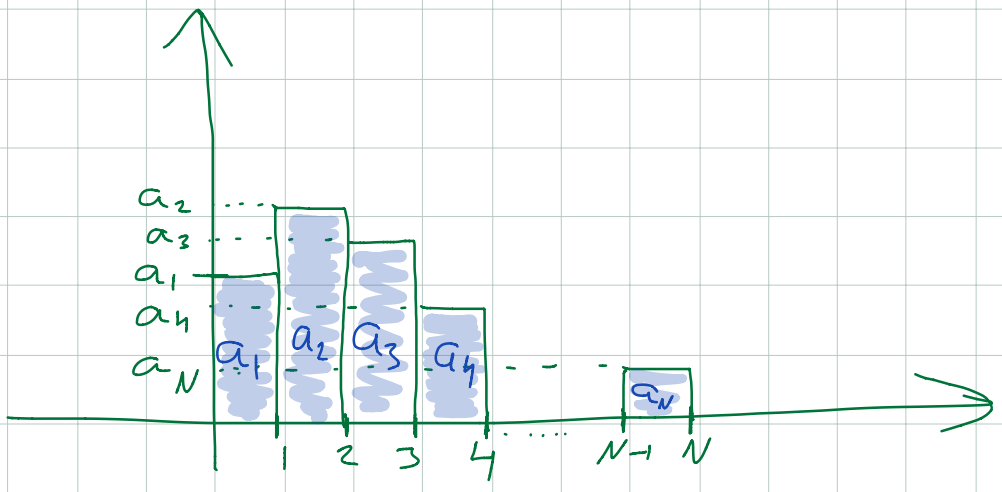


10.3: Convergence of Series with positive terms

Let $a_n \geq 0$ for all n

Idea: $S_N = a_1 + a_2 + a_3 + \cdots + a_N$

Think of these as "areas of rectangles of height a_n & width 1."



So S_N is the area of the first N rectangles and S_{N+1} is the area of the first $N+1$ rectangles

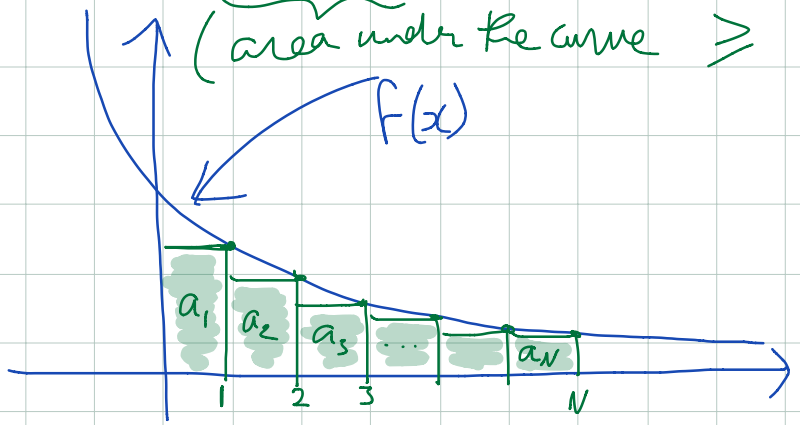
$\Rightarrow$ $S_N \leq S_{N+1}$

Integral Test

Now, let $a_n = f(n)$ where $\begin{cases} f(x) \geq 0 \\ f(x) \text{ decreasing} \\ f(x) \text{ continuous} \end{cases}$

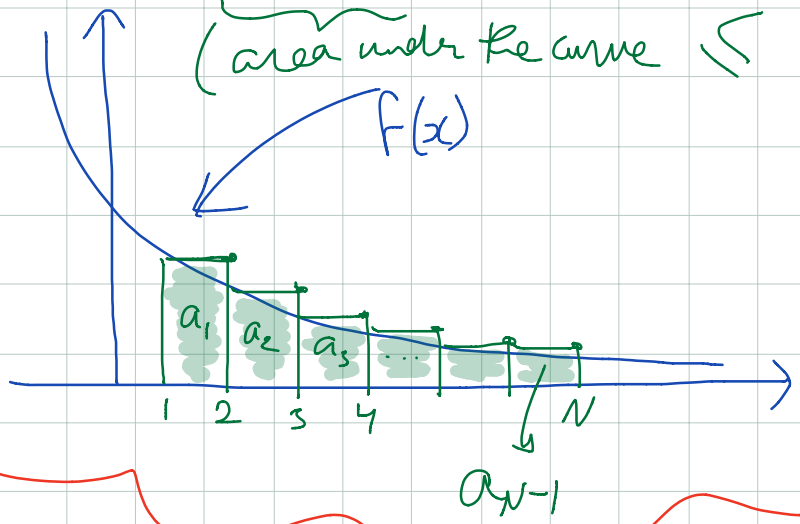
(i) If $\int_1^\infty f(x)$ converges, then $\sum_{n=1}^{\infty} a_N$ converges

(area under the curve $\geq$ area of rectangles)



(ii) If $\int_1^\infty f(x)$ diverges, then $\sum_{n=1}^{\infty} a_N$ diverges

(area under the curve $\leq$ area of rectangles)



Example: Does the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ converge or diverge?

Soln: Let $f(x) = \frac{1}{x}$, which is positive continuous & decreasing

Moreover: $\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{x}\,dx = \infty$ (why?)

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the integral test.

Theorem: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ & diverges otherwise

e.g. $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges

e.g. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ diverges

Example: Does $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$ converge?

Solution: Let $f(x) = \frac{x}{(x^2+1)^2}$ $f(x) \geq 0$

$f(x)$ continuous

$f(x)$ decreasing?

yes! $f'(x) = \frac{(x^2+1)^2 - 2x(x^2+1)x}{(x^2+1)^4} = \frac{(x^2+1)(1-3x^2)}{(x^2+1)^4} < 0$ for $x \geq 1$

Moreover $\int_1^\infty \frac{x}{(x^2+1)^2}\,dx \overset{u=x^2+1}{=} \cdots = \frac{1}{4}$

$\Rightarrow \sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$ converges by the integral test.

Theorem: (Comparison Test)

Assume that $0 \le a_n \le b_n$ for all $n \ge M$ (some number)

(i) If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges

(ii) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges

Example: Does $\sum_{n=1}^{\infty} \frac{(0.9)^n}{\sqrt{n}}$ converge?

Soln: $\underbrace{\frac{0.9^n}{\sqrt{n}}}_{a_n} \le \underbrace{0.9^n}_{b_n}$ for all $n \ge 1$

and

$\sum_{n=1}^{\infty} 0.9^n$ converges (Geometric Series)

So $\sum_{n=1}^{\infty} \frac{0.9^n}{\sqrt{n}}$ converges by the comparison test.

Example: Does $\sum_{n=1}^{\infty} \frac{1}{(n^2+7)^{1/3}}$ converge?

Sol'n Idea: $\sum_{n=1}^{\infty} \frac{1}{(n^2+7)^{1/3}}$ "feels" like a p-series with $p<1$

Rigorously: It's enough to show that $\frac{1}{(n^2+7)^{1/3}} \geq \frac{1}{n}$ for $n \geq M$ ← some number

(bec $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so we'd be done by the comparison test)

So let's show that $\frac{1}{(n^2+7)^{1/3}} \geq \frac{1}{n}$

This is the same as showing that $(n^2+7)^{1/3} \leq n$

which is the same as $n^2+7 \leq n^3$

and $n^3-(n^2+7) \geq 0$

but this is true for all $n \geq 3$ (why?)

Hint: graph $f(n)=n^3$ & $g(n)=n^2+7$

So $\frac{1}{n} \leq \left(\frac{1}{n^2+7}\right)^{1/3} \Rightarrow$ the series diverges by the comparison test.

Limit Comparison Test

Let $a_n \geq 0$ & $b_n \geq 0$ for all n

and suppose that the limit

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} \text{ exists}$$

Then

(1) If $L > 0$ ($\sum a_n$ converges if and only if $\sum b_n$ converges)

(2) If $L = \infty$ & $\sum a_n$ converges then

$\sum b_n$ converges

(3) If $L = 0$ & $\sum b_n$ converges then

$\sum a_n$ converges

Example: Let's revisit $\underbrace{\sum_{n=1}^{\infty} \frac{1}{(n^2+7)^{1/3}}}_{a_n}$

Define $b_n = \frac{1}{(n^2)^{1/3}} = \frac{1}{n^{2/3}}$

and note that $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{(n^2+7)^{1/3}}}{\frac{1}{(n^2)^{1/3}}}$

$= \lim_{n\to\infty} \frac{n^{2/3}}{(n^2+7)^{2/3}} = \lim_{n\to\infty} \left(\frac{n^2}{n^2+7}\right)^{1/3} = 1$

Moreover, $\sum \frac{1}{n^{2/3}}$ diverges (p-series with $p<1$)

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{(n^2+7)^{1/3}}$ diverges by the Limit test.

10.4 Absolute Convergence

We say that $\sum a_n$ converges absolutely if $\sum |a_n|$ converges

Theorem: $\sum |a_n|$ converges $\Rightarrow \sum a_n$ converges

Example:

$$\sum_{n=1}^{\infty} (-1)^{n-1}/n^2 \quad \text{converges (by the theorem)}$$

because $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

convergent p-series with $p=2$